Math 201 — Fall 2011–12
Calculus and Analytic Geometry III, all sections
Quiz 2, November 19 — Duration: 80 minutes

GRADES:

1 (/21)	2 (/12)	3 (/16)	4 (/12)	5 (/15)	6 (/12)	7 (/12)	TOTAL	GRADE

YOUR NAME: Key

YOUR AUB ID#: ____

PLEASE CIRCLE YOUR SECTION:

Section 1 MWF 3, Kobeissi Recitation F 11	Section 2 MWF 3, Kobeissi Recitation F 5	Section 3 MWF 3, Kobeissi Recitation F 4	Section 4 MWF 3, Kobeissi Recitation F 10
Section 5 MWF 10, Abi-Khuzam Recitation T 11	Section 6 MWF 10, Abi-Khuzam Recitation T 3:30	Section 7 MWF 10, Abi-Khuzam Recitation T 5	Section 8 MWF 10, Abi-Khuzam Recitation T 2
Section 9 MWF 11, Brock Recitation T 12:30	Section 10 MWF 11, Brock Recitation T 2	Section 11 MWF 11, Brock Recitation T 11	Section 12 MWF 11, Brock Recitation T 3:30
Section 13 MWF 2, Nahlus Recitation Th 11	Section 14 MWF 2, Nahlus Recitation Th 3:30	Section 15 MWF 2, Nahlus Recitation Th 8	Section 16 MWF 2, Nahlus Recitation Th 5
Section 17 MWF 8, Makdisi Recitation F 2	Section 18 MWF 8, Makdisi Recitation Th 8	Section 19 MWF 8, Makdisi Recitation Th 2	Section 20 MWF 8, Makdisi Recitation Th 3:30
Section 21 MWF 1, Raji Recitation M 8	Section 22 MWF 1, Raji Recitation M 9	Section 23 MWF 1, Raji Recitation M 4	
Section 24 MWF 10, Egeileh Recitation F 11	Section 25 MWF 10, Egeileh Recitation F 2	Section 26 MWF 10, Egeileh Recitation F 3	

INSTRUCTIONS:

1. Write your NAME and AUB ID number, and circle your SECTION above.
2. Solve the problems inside this white booklet. Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit.
3. You may also use the back of any page for solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.
4. Closed book and notes. NO CALCULATORS ALLOWED. Turn OFF and put away any cell phones.

GOOD LUCK!

1. **(7 pts each part, 21 pts total)** Given the level surface S of equation

$$x^2 - y^2 - z^2 = -4.$$

(a) Find an equation for the tangent plane to the surface S at the point $P(3, 2, -3)$.

S is a level surface for the function $f(x,y,z) = x^2 - y^2 - z^2$.

$\vec{\nabla} f(x,y,z) = 2x\vec{i} - 2y\vec{j} - 2z\vec{k}$

$\vec{\nabla} f(3,2,-3) = 6\vec{i} - 4\vec{j} + 6\vec{k}$ normal vector to tangent plane at $(3,2,-3)$.

$\therefore$ Equation of tangent plane at $P(3,2,-3)$ is:

$6(x-3) - 4(y-2) + 6(z+3) = 0$

$6x - 4y + 6z + 8 = 0.$

(b) Find parametric equations of the normal line to the surface S at $P(3, 2, -3)$.

From part (a), $\vec{\nabla} f(3,2,-3) = 6\vec{i} - 4\vec{j} + 6\vec{k}$, and this is a vector parallel to the normal line at $(3,2,-3)$.

$\therefore$ The parametric equations of the normal line through P are:

$x - 3 = 6t$

$y - 2 = -4t$

$z + 3 = 6t, \quad -\infty < t < \infty.$

(c) Does the normal line in part (b) intersect the surface S in a point other than P? If yes, find the coordinates of the other point of intersection.

The normal line intersects S if and only if the point $(3+6t, 2-4t, -3+6t)$ lies on the surface S for some value of t. i.e

$(3+6t)^2 - (2-4t)^2 - (-3+6t)^2 = -4,$

$(9+36t+36t^2) - (4-16t+16t^2) - (9-36t+36t^2) = -4,$

$-16t^2 + 88t = 0, \quad t(-16t+88) = 0.$

The roots are $t = 0$ which gives the point $P(3,2,-3)$,

and $t = \frac{11}{2}$ " " " " $Q(3+33, 2-22, -3+33)$.

So the normal line does intersect the surface S at another point $Q(36, -20, 30)$.

2. **(6 pts each part, 12 pts total)** Given the two level surfaces

$$S_1 = \{(x,y,z) : x^2+y^2+z^2 = 4\}, \text{ and } S_2 = \{(x,y,z) : x+y=0\}$$

(a) Find a vector normal to the surface S_1 at the point $P(1,-1,\sqrt{2})$. And find a second vector normal to the surface S_2 at the same point $P(1,-1,\sqrt{2})$.

S_1 and S_2 are level surfaces for the functions $f(x,y,z) = x^2+y^2+z^2$ and $g(x,y,z) = x+y$, respectively.

$\vec{\nabla} f(x,y,z) = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$, $\quad \vec{\nabla} g(x,y,z) = \vec{i} + \vec{j}$.

$\vec{\nabla} f(1,-1,\sqrt{2}) = 2\vec{i} - 2\vec{j} + 2\sqrt{2}\vec{k}$, $\quad \vec{\nabla} g(1,-1,\sqrt{2}) = \vec{i} + \vec{j}$.

Thus $\vec{\nabla} f(1,-1,\sqrt{2}) = 2\vec{i} - 2\vec{j} + 2\sqrt{2}\vec{k}$ is normal to S_1 at P,

and $\vec{\nabla} g(1,-1,\sqrt{2}) = \vec{i} + \vec{j}$ " " " S_2 at P.

(b) Find parametric equations for the line that is tangent to the curve of intersection of the two surfaces S_1, and S_2 at the same point $P(1,-1,\sqrt{2})$.

A vector $\vec{v}$ parallel to the required tangent line is

$$\vec{v} = \vec{\nabla} f(P) \times \vec{\nabla} g(P) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 2\sqrt{2} \\ 1 & 1 & 0 \end{vmatrix}$$

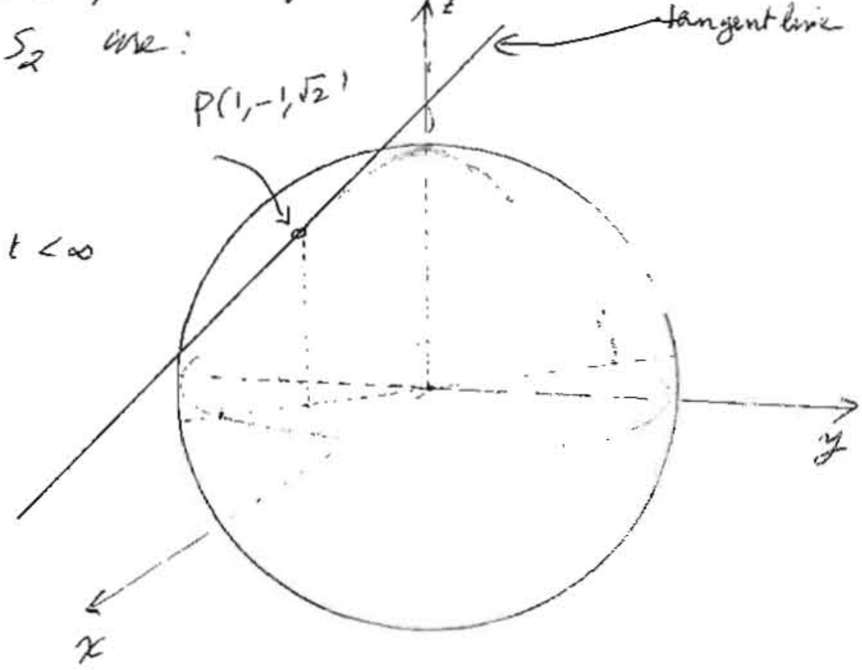
$$= -2\sqrt{2}\,\vec{i} + 2\sqrt{2}\,\vec{j} + 4\vec{k}$$

$\therefore$ parametric equations of the tangent line to the curve of intersection of S_1 and S_2 are:

$x - 1 = -2\sqrt{2}\,t,$

$y + 1 = 2\sqrt{2}\,t,$

$z - \sqrt{2} = 4t, \quad -\infty < t < \infty$



3. **(8 pts each part, 16 pts total)** Given a function $f(x, y)$ which satisfies

$$f(2,1) = 10, \qquad \nabla f|_{(2,1)} = 4\mathbf{i}+3\mathbf{j}, \qquad \nabla f|_{(4,3)} = 6\mathbf{i}+5\mathbf{j}, \qquad \nabla f|_{(6,5)} = 2\mathbf{i}+\mathbf{j}.$$

(a) Find the directional derivative of f at the point $P(6,5)$ in the direction of the vector $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$.

From $\vec{\nabla} f(6,5) = 2\vec{i} + \vec{j}$, we get $f_x(6,5) = 2$, $f_y(6,5) = 1$.

A unit vector in the direction of the vector $\vec{v}$ is given by:

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{4\vec{i}+3\vec{j}}{\sqrt{16+9}} = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}.$$

$\therefore$ The directional derivative of f at $P(6,5)$ in the direction of $\vec{v}$ is

$$D_{\vec{u}} f(6,5) = \vec{\nabla} f(6,5) \cdot \vec{u} = (2\vec{i}+\vec{j}) \cdot \left(\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}\right)$$
$$= \frac{8}{5} + \frac{3}{5} = \frac{11}{5}.$$

(b) Find an approximate value of $f(1.99, 1.02)$.

From $\vec{\nabla} f(2,1) = 4\vec{i} + 3\vec{j}$ we get $f_x(2,1) = 4$, $f_y(2,1) = 3$.

Now, ~~by~~ from the mean-value theorem, or using linear approximation,

$$f(1.99, 1.02) - f(2,1) \approx (1.99-2) f_x(2,1) + (1.02-1) f_y(2,1)$$

$$= (-0.01)4 + (0.02)3 = -0.04 + 0.06 = 0.02.$$

$$\therefore \quad f(1.99, 1.02) \approx f(2,1) + 0.02 = 10 + 0.02 = 10.02.$$

4. **(12 pts)** Use the method of Lagrange multipliers to find the maximum value of $f(x,y) = x^2 + y^2 - 3x - 2y$ on the curve $x^2 + y^2 = 13$.

We start from the equation $\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$, where $g(x,y) = x^2+y^2-13$. We obtain

$$\begin{cases} (2x-3)\vec{i} + (2y-2)\vec{j} = \lambda(2x\vec{i} + 2y\vec{j}), & \text{along with} \\ g(x,y) = 0 \end{cases}$$

This gives us the system

$$\begin{cases} 2x-3 = 2\lambda x \\ 2y-2 = 2\lambda y \\ x^2+y^2 = 13 \end{cases}$$

or $$\begin{cases} 2x(1-\lambda) = 3 \\ 2y(1-\lambda) = 2 \\ x^2+y^2 = 13 . \end{cases}$$

Clearly, then $\lambda \neq 1$, and so $x = \frac{3}{2(1-\lambda)}$, $y = \frac{1}{1-\lambda}$.

Using these in the third equation we get

$$\left(\frac{3}{2(1-\lambda)}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 = 13, \quad \frac{9}{4(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 13,$$

$$\frac{13}{4(1-\lambda)^2} = 13, \quad (1-\lambda)^2 = \frac{1}{4}, \quad 1-\lambda = \pm\frac{1}{2}.$$

$$\therefore \quad x = \frac{3}{2(\pm\frac{1}{2})}, \quad y = \frac{1}{\pm 1/2}.$$

$$x = \pm 3, \quad y = \pm 2.$$

Since x and y have the same sign, the possible extreme points are:

$(3,2)$ or $(-3,-2)$.

Then $f(3,2) = 13-9-4 = 0$

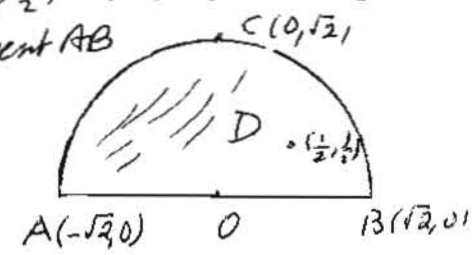
$f(-3,-2) = 13+9+4 = 26.$

$\therefore$ The maximum value of f on the curve is $\underline{26}$.

5. **(15 pts)** Find the maximum and minimum values of the function $f(x, y) = x^2 + y^2 - x - y$ on $D = \{(x, y) : x^2 + y^2 \leq 2, \quad y \geq 0\}$.

Critical points are obtained from the system $\begin{matrix} 2x-1=0 \\ 2y-1=0 \end{matrix}$, $\begin{matrix} x=\frac{1}{2} \\ y=\frac{1}{2} \end{matrix}$.

So there is one critical point of f, namely $(\frac{1}{2}, \frac{1}{2})$ which is in D.
The boundary of D consists of two parts, segment AB and arc $\widehat{ACB}$.



Boundary AB:

$y=0$, $-\sqrt{2} \leq x \leq \sqrt{2}$, $f(x,0) = x^2 - x$,

$\frac{d}{dx} f(x,0) = 2x - 1$; derivative vanishes at $x = \frac{1}{2}$

so possible extremum points on AB are $(-\sqrt{2},0)$, $(\frac{1}{2},0)$, $(\sqrt{2},0)$.

Boundary $\widehat{ACB}$.

$y = \sqrt{2-x^2}$, $-\sqrt{2} \leq x \leq \sqrt{2}$.

$f(x, \sqrt{2-x^2}) = 2 - x - \sqrt{2-x^2}$.

derivative $= -1 + \frac{x}{\sqrt{2-x^2}}$. So $-1 + \frac{x}{\sqrt{2-x^2}} = 0$ gives $\frac{x}{\sqrt{2-x^2}} = 1$;

so x is positive and $x^2 = 2 - x^2$, $x^2 = 1$, $x = \pm 1$. Hence $x = 1$, and the possible extremum points on arc ACB are

$(-\sqrt{2}, 0)$, $(1,1)$, $(\sqrt{2}, 0)$.

Now we compute the values of $f(x,y)$ at all the possible extremum points:

$f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2}$.

$f(-\sqrt{2}, 0) = 2 + \sqrt{2}$.

$f(\frac{1}{2}, 0) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$.

$f(\sqrt{2}, 0) = 2 - \sqrt{2}$.

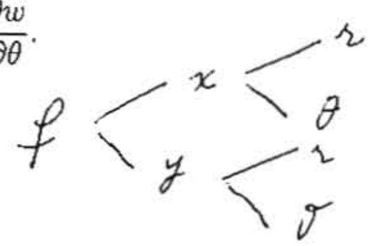
$f(1,1) = 1 + 1 - 1 - 1 = 0$.

Hence,

the maximum value of f on D is $2 + \sqrt{2}$.
the minimum " " " " " $-\frac{1}{2}$.

6. **(6 pts each part, total 12 pts)**

(a) If $w = f(x,y)$, and $x = r\cos\theta, y = r\sin\theta$, use the chain rule to compute $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$.



Using the chain rule,

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} = f_x \cos\theta + f_y \cdot \sin\theta .$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} = f_x\cdot(-r\sin\theta) + f_y\cdot(r\cos\theta) .$$

(b) Compute $\frac{\partial^2 w}{\partial r^2}$ and simplify your answer.

From part (a),

$$\frac{\partial w}{\partial r} = f_x\cdot\cos\theta + f_y\cdot\sin\theta .$$

$$\therefore \quad \frac{\partial^2 w}{\partial r^2} = \cos\theta\left(f_{xx}\cos\theta + f_{xy}\sin\theta\right) + \sin\theta\left(f_{yx}\cos\theta + f_{yy}\sin\theta\right)$$

$$= (\cos^2\theta) f_{xx} + \sin\theta\cos\theta\,(f_{xy} + f_{yx}) + (\sin^2(\theta)) f_{yy}$$

If the partial derivatives are continuous, then the mixed partials are equal and we can simplify to

$$(\cos^2\theta) f_{xx} + 2(\sin\theta\cos\theta) f_{xy} + (\sin^2\theta) f_{yy} .$$

7. **(6 pts each part, 12 pts total)** Consider the function

$$f(x,y)=\begin{cases}\dfrac{x^2y}{2x^2+3\sin^2 y}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0).\end{cases}$$

(a) Use the definition of differentiability to prove that f is **NOT** differentiable at $(0,0)$.

For this part, note that $f(0,0)=0$. You may use without proof that $f_x(0,0)=0$ and $f_y(0,0)=0$.

We need to consider

$$\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$=\lim_{(h,k)\to(0,0)} \frac{h^2k}{2h^2+3\sin^2 k}\cdot\frac{1}{\sqrt{h^2+k^2}}.$$

If $h=0$, and $k\to 0$, the corresponding limit is 0.

If $k=h$, and $h\to 0$, we get, say if $h>0$,

$$\lim_{\substack{h\to 0\\ h>0}} \frac{h^3}{2h^2+3\sin^2 h}\cdot\frac{1}{\sqrt{2}|h|}=\lim_{\substack{h\to 0\\ h>0}}\frac{1}{\left[2+3\left(\frac{\sin h}{h}\right)^2\right]\sqrt{2}}=\frac{1}{(2+3)\sqrt{2}}=\frac{1}{5\sqrt{2}}$$

Hence the limit considered does not exist and f is not differentiable at $(0,0)$.

(b) Prove that f is continuous at $(0,0)$.

Since $\sin^2 y\geq 0$, we always have

$$0\leq x^2\leq 2x^2+3\sin^2 y.$$

So, if $(x,y)\neq(0,0)$, then $2x^2+3\sin^2 y\neq 0$,

$$|f(x,y)|=\frac{x^2|y|}{2x^2+3\sin^2 y}\leq\frac{(2x^2+3\sin^2 y)|y|}{2x^2+3\sin^2 y}=|y|$$

This implies that

$$\lim_{(x,y)\to(0,0)} f(x,y)=0$$

But $f(0,0)=0$, so

$$\lim_{(x,y)\to(0,0)} f(x,y)=f(0,0),$$

hence f is continuous at $(0,0)$.